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Research Paper

Commutative power-associative representations of symmetric matrices

Lucia S.I. Murakami [a,*], Pablo S.M. Nascimento [b], Ivan Shestakov [a], Juaci Picanço da Silva [c]

[a] *Department of Mathematics - IME, University of São Paulo, Rua do Matão 1010, São Paulo, SP, 05508–090, Brazil*
[b] *Faculty of Mathematics, UNIFESSPA, Folha 31, Quadra 07, Lote Especial, s/n, Marabá, PA, 68507-590, Brazil*
[c] *Faculty of Mathematics - ICEN, Federal University of Pará, Rua Augusto Corrêa, 01 - Guamá, CEP 66075-110, Belém, PA, Brazil*

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A B S T R A C T

The classification of irreducible unital commutative power-associative modules for $H_n(F)$, the algebra of symmetric matrices with the Jordan product, over a field F of characteristic not 2, 3 and 5 are given, for $n \geq 3$. It is proved that there exists, up to isomorphisms, only one irreducible module which is not Jordan. It is also shown that every finite dimensional unital commutative power associative module for this algebra is completely reducible.

* Corresponding author.
E-mail addresses: ikemoto@ime.usp.br (L.S.I. Murakami), pablosalermo@unifesspa.edu.br (P.S.M. Nascimento), shestak@ime.usp.br (I. Shestakov), jps@ufpa.br (J. Picanço da Silva).

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1. Introduction

In this paper, F will be a field of characteristic different from 2, 3 and 5 and algebras will be considered over this field. Recall that an algebra A is *power-associative* if every subalgebra of A generated by a single element is associative.

The class of commutative power-associative algebras is widely studied since the works of A.A. Albert ([1], [2]). This class includes all Jordan algebras and there are some results connecting these two classes of algebras. Albert ([2]) defined the *degree* of a simple commutative power-associative algebra A to be the maximum number of mutually orthogonal idempotents in A_K, where K is the algebraic closure of the center of A, and proved that if A has degree greater than 2, then A is a Jordan algebra. If A has degree 2, there exist simple algebras which are not Jordan, if the characteristic of F is $p > 5$ ([3], [9]). In characteristic zero, Kokoris ([10]) proved that every semisimple commutative power-associative algebra is a Jordan algebra.

Another question connecting these two classes was asked by Albert ([1]): for Jordan algebras, it is true that finite dimensional nil algebras are nilpotent; is the same true for commutative power-associative algebras? The answer is negative. D. Suttles showed that there exists a 5-dimensional commutative power-associative nil algebra which is not nilpotent ([13]). Since then, many authors have investigated the validity of the following question, known as Albert's Problem: "Is every finite dimensional commutative power-associative nilalgebra solvable?" For some particular cases, there are positive answers for this question. See [4] and [11] for a status of the solved cases. Related to Albert's Problem, representations of commutative power-associative nil algebras have also been studied. See, for instance [5] and [6].

The aim of this paper is to investigate the structure of unital commutative power-associative modules for the algebra $H_n(F)$ of symmetric $n \times n$ matrices over the field F, with the Jordan product, for $n \geq 3$. It is well known that $H_n(F)$ is a simple Jordan algebra and the structure of unital Jordan modules for this algebra was obtained by N. Jacobson in [7] by means of describing the structure of the unital enveloping algebra of $H_n(F)$. We reobtain this classification in a direct way as a consequence of the main theorem of this paper (Theorem 8.1): we classify the irreducible unital commutative power-associative modules for $H_n(F)$ and prove that there is, up to isomorphisms, only one which is not Jordan. Moreover, we show that every finite dimensional unital module M for this algebra is completely reducible.

The strategy to prove the main theorem consists in considering the Peirce Decomposition of the module M relative to a complete set of orthogonal idempotents of $H_n(F)$ and choosing a convenient basis for the component M_{12}. In this basis, each element will generate a submodule isomorphic to an irreducible Jordan module. The non Jordan component will be the ideal B considered by Schafer ([12]) for the split null extension $A = H_n(F) \oplus M$.

In Section 2 we state the known results concerning the structure of commutative power-associative algebras which will be needed here. Section 3 is devoted to present examples of irreducible unital modules for $H_n(F)$. They will be shown to be a complete list of irreducible unital modules for this algebra. In Section 4 we prove general results for $H_n(F)$-modules which will be used in Sections 5 and 6, in order to identify the generators of the irreducible Jordan submodules of an arbitrary commutative power-associative unital module M. In Section 7 we find out the non Jordan submodules of M. Finally, in Section 8, we prove that M is completely reducible and that examples of Section 3 are the only possible irreducible components of M.

2. Preliminaries

It is well known that an algebra A is commutative and power-associative if and only if A satisfies the following identity

$$x^2x^2 - x(xx^2) = 0,$$

which is equivalent to its linearization

$$[x, y, z, w] = 0, \tag{2.1}$$

where

$$\begin{aligned}[x, y, z, w] = {} & 4\big((xy)(zw) + (xz)(yw) + (xw)(yz)\big) \\ & - x\big(y(zw) + z(yw) + w(yz)\big) - y\big(x(zw) + z(xw) + w(xz)\big) \\ & - z\big(x(yw) + y(xw) + w(xy)\big) - w\big(x(yz) + y(xz) + z(xy)\big).\end{aligned}$$

Recall that a F-vector space M with left and right actions $\cdot$ of A on M is a commutative power-associative bimodule for A if the split null extension $A + M$ is a commutative power-associative algebra. In particular, $a \cdot m = m \cdot a$, for $a \in A$ and $m \in M$. For this reason, we will refer to these bimodules as (left) modules. It is straightforward to see that M is a commutative power-associative module for A if, and only if,

$$4a^2 \cdot (a \cdot m) = a^3 \cdot m + a \cdot (a^2 \cdot m) + 2a \cdot \big(a \cdot (a \cdot m)\big), \tag{2.2}$$

for $a \in A$ and $m \in M$.

Let A be a commutative power-associative algebra and let $e \in A$ an idempotent. The *Peirce Decomposition* of A relative to this idempotent is

$$A = A_e(1) \oplus A_e(\tfrac{1}{2}) \oplus A_e(0),$$

where $A_e(\lambda) = \{x \in A : ex = \lambda x\}$, for $\lambda = 0, 1, \frac{1}{2}$. These subspaces satisfy the following relations

$$
\begin{aligned}
A_e(0)^2 &\subseteq A_e(0),\\
A_e(\tfrac{1}{2})^2 &\subseteq A_e(0) \oplus A_e(1),\\
A_e(1)^2 &\subseteq A_e(1),\\
A_e(0)A_e(\tfrac{1}{2}) &\subseteq A_e(\tfrac{1}{2}) \oplus A_e(1),\\
A_e(0)A_e(1) &= 0,\\
A_e(\tfrac{1}{2})A_e(1) &\subseteq A_e(0) \oplus A_e(\tfrac{1}{2}).
\end{aligned}
$$

For $a \in A$ and $\lambda \in \{0, 1, \frac{1}{2}\}$, we denote by a_λ the component of a in the subspace $A_e(\lambda)$. As proved in [2], we have, for each $x \in A_e(1)$, a linear mapping $S_{\frac{1}{2}}(x)\colon A_e(\frac{1}{2}) \to A_e(\frac{1}{2})$ defined by $S_{\frac{1}{2}}(x)(y) = (xy)_{\frac{1}{2}}$, for $y \in A_e(\frac{1}{2})$. Moreover, $S\colon A_e(1) \to \left[\mathrm{End}_F\left(A_e(\frac{1}{2})\right)\right]^+$, which sends $x \in A_e(1)$ to $2S_{\frac{1}{2}}(x)$, is a homomorphism of Jordan algebras. Therefore,

$$
B_e = \ker S = \{b \in A_e(1) : bA_e(\tfrac{1}{2}) \subseteq A_e(0)\} \tag{2.3}
$$

is an ideal of $A_e(1)$ and $A_e(1)/B_e$ is a Jordan algebra.

Let A be commutative power-associative algebra with unity 1 and let $e_1, \ldots, e_n$, $n \geq 3$, be a set of pairwise orthogonal idempotents such that $1 = e_1 + \cdots + e_n$. Then the Peirce Decomposition relative to this set of idempotents is

$$
A = \bigoplus_{1 \leq i \leq j \leq n} A_{ij}, \text{ where } A_{ii} = A_{e_i}(1), A_{ij} = A_{e_i}(\tfrac{1}{2}) \cap A_{e_j}(\tfrac{1}{2}), \text{ if } i < j.
$$

If $j > i$ we denote $A_{ji} = A_{ij}$. These subspaces satisfy

$$
\begin{aligned}
A_{ii}^2 &\subset A_{ii},\\
A_{ii}A_{ij} &\subset A_{ij} + A_{jj} \quad (i \neq j),\\
A_{ij}^2 &\subset A_{ii} + A_{jj} \quad (i \neq j),\\
A_{ij}A_{jk} &\subset A_{ik} \quad (i, j, k \text{ distinct}),\\
A_{ij}A_{kl} &= 0 \quad (k \neq i, j \text{ and } l \neq i, j).
\end{aligned} \tag{2.4}
$$

3. Examples of irreducible modules

Let $\mathcal{J}$ be a Jordan algebra with 1. Recall that its universal multiplication envelope $\mathcal{U}(\mathcal{J})$ decomposes as

$$
\mathcal{U}(\mathcal{J}) = \mathcal{U}_0(\mathcal{J}) \oplus \mathcal{U}_{\frac{1}{2}}(\mathcal{J}) \oplus \mathcal{U}_1(\mathcal{J}),
$$

where $\mathcal{U}_1(\mathcal{J})$ is the universal unital multiplication envelope of $\mathcal{J}$ and $\mathcal{U}_0(\mathcal{J}) \oplus \mathcal{U}_{\frac{1}{2}}(\mathcal{J})$ is canonically isomorphic to $S(\mathcal{J})$, the special universal envelope of $\mathcal{J}$, via $\sigma \mapsto 2\sigma$ (see

[8, Chapter 2]). Moreover, this correspondence maps $\mathcal{U}_{\frac{1}{2}}(\mathcal{J})$, the universal envelope for semi-unital representations, to $S_1(\mathcal{J})$, the unital special universal envelope of $\mathcal{J}$.

Let A be an associative algebra with 1 and suppose that $\mathcal{J}$ is a subalgebra of the Jordan algebra $A^{(+)}$. Let M be an associative unital left A-module, with the action of $a \in A$ on $m \in M$ be denoted by am. Then, defining

$$a \cdot m = m \cdot a = \tfrac{1}{2}am, \quad \text{for } a \in \mathcal{J}, m \in M,$$

M became a $\mathcal{J}$-Jordan module, which corresponds to a semi-unital representation σ. By the above correspondence, 2σ is a unital associative specialization. In general, 2σ is not a Jordan representation. In fact, if 2σ is a Jordan representation, we must have

$$(xyx - x^2 \circ y)m = 0, \quad \text{for all } x, y \in \mathcal{J}, m \in M.$$

For instance if $A = M_n(D)$, with $n > 1$ and D any division algebra, taking $x = e_{11}$ and $y = e_{12} + e_{21}$, we should have $y \cdot m = 0$, for every $m \in M$. Therefore, any irreducible left A-module could not be a Jordan $\mathcal{J}$-module with the above action. However, 2σ is a representation of $\mathcal{J}$ as a commutative power-associative algebra, since the new action is given by

$$a \cdot m = m \cdot a = am, \quad \text{for } a \in \mathcal{J}, m \in M,$$

and (2.2) is clearly satisfied by any $a \in \mathcal{J}$ and $m \in M$, by the associativity of the original action of A on M.

In this section we present the relation of irreducible unital commutative power-associative modules for $H_n(F)$, for $n \geq 3$ which can be obtained from the structure of Jordan modules. Recall that the multiplication in $H_n(F)$ is the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ where xy denotes the usual matrix multiplication. A linear basis for $H_n(F)$ is

$$e_i = e_{ii} \quad (i = 1, \ldots, n), \qquad u_{ij} = e_{ij} + e_{ji} \quad (1 \leq i < j \leq n), \tag{3.1}$$

where e_{ij} stands for the elementary matrix which has 1 in the entry (i, j) and 0 otherwise. We also denote $u_{ji} = u_{ij}$, for $i < j$.

Example 3.1. It is well known that the $H_n(F)$-regular module is an irreducible Jordan module. We denote this module by $\overline{H_n(F)}$ and, for $a \in H_n(F)$, the corresponding element in $\overline{H_n(F)}$ will be denoted by $\overline{a}$. Therefore, the action of $H_n(F)$ in $\overline{H_n(F)}$ satisfies, for distinct i, j, k, l,

$$\begin{gathered} e_i \cdot \overline{e_i} = \overline{e_i},\ e_i \cdot \overline{e_j} = 0,\ e_i \cdot \overline{u_{jk}} = 0,\ e_i \cdot \overline{u_{ij}} = \tfrac{1}{2}\overline{u_{ij}}, \\ u_{ij} \cdot \overline{u_{ij}} = \overline{e_i} + \overline{e_j},\ u_{ij} \cdot \overline{u_{jk}} = \tfrac{1}{2}\overline{u_{ik}},\ u_{ij} \cdot \overline{u_{kl}} = 0, \\ u_{ij} \cdot \overline{e_i} = \tfrac{1}{2}\overline{u_{ij}},\ u_{ij} \cdot \overline{e_k} = 0. \end{gathered} \tag{3.2}$$

Example 3.2. It is also known that $S_n(F)$, the set of skew-symmetric matrices, is an irreducible Jordan module for $H_n(F)$. The action of $H_n(F)$ on $S_n(F)$ is also by the Jordan product, that is

$$a \cdot v = v \cdot a = \tfrac{1}{2}(av + va), \quad \text{for } a \in H_n(F), v \in S_n(F).$$

Therefore, $S_n(F)$ is an example of irreducible commutative power-associative module for $H_n(F)$. If $\mathbf{v}_{ij} = e_{ij} - e_{ji}$, for $1 \le i, j \le n, i \ne j$, then $\mathbf{v}_{ji} = -\mathbf{v}_{ij}$ and $\{\mathbf{v}_{ij} : 1 \le i < j \le n\}$ is a linear basis of $S_n(F)$. The action of the canonical basis of $H_n(F)$ given in (3.1) on the elements $\mathbf{v}_{ij}$ is

$$e_i \cdot \mathbf{v}_{ij} = \tfrac{1}{2}\mathbf{v}_{ij},\ e_k \cdot \mathbf{v}_{ij} = 0,\ u_{ij} \cdot \mathbf{v}_{ij} = 0,\ u_{ij} \cdot \mathbf{v}_{jk} = \tfrac{1}{2}\mathbf{v}_{ik},\ u_{ij} \cdot \mathbf{v}_{kl} = 0, \tag{3.3}$$

for distinct i, j, k, l.

Example 3.3. Recall that the unique irreducible unital $M_n(F)$-associative left module is the set $M_{n\times 1}(F)$ of $n \times 1$ matrices with entries in F with the action given by matrix multiplication. Therefore,

$$m \cdot a = a \cdot m = am, \quad \text{for } a \in H_n(F) \text{ and } m \in M_{n\times 1}(F),$$

where am stands for the usual matrix multiplication, defines a commutative power-associative $H_n(F)$-module structure on $M_{n\times 1}(F)$. This module is not Jordan as noted before. The action of $H_n(F)$ on the basis $\{m_1, \dots, m_n\}$ of $M_{n\times 1}(F)$,

$$e_i \cdot m_i = m_i,\ e_j \cdot m_i = 0,\ u_{ij} \cdot m_j = m_i,\ u_{ik} \cdot m_j = 0, \tag{3.4}$$

for distinct $i, j, k \in \{1, \dots, n\}$ will also be used.

In next sections we shall prove that $\overline{H_n(F)}$, $S_n(F)$ and $M_{n\times 1}(F)$ are the only irreducible commutative power-associative modules for $H_n(F)$ and every finite dimensional commutative power-associative module for $H_n(F)$ is completely reducible.

4. Commutative power-associative modules

From now on, let M be a commutative power-associative module for $H_n(F)$ and consider $A = H_n(F) + M$ the split null extension of M, as described above. The action of an element $a \in H_n(F)$ on $m \in M$ will be denoted simply by am. Also, the Jordan product symbol in $H_n(F)$ will be omitted. Therefore, the multiplication in $H_n(F)$ is, for distinct i, j, k, l,

$$\begin{gathered} e_i^2 = e_i;\ e_i e_j = 0;\ e_i u_{ij} = \tfrac{1}{2}u_{ij};\ e_i u_{jk} = 0; \\ u_{ij}^2 = e_i + e_j;\ u_{ij}u_{jk} = \tfrac{1}{2}u_{ik};\ u_{ij}u_{kl} = 0. \end{gathered} \tag{4.1}$$

Recall that $e_1, \dots, e_n$ form a set of orthogonal idempotents and $1 = e_1 + \cdots + e_n$. Let $A = \bigoplus_{1 \le i \le j \le n} A_{ij}$ be the Peirce decomposition of A for this set of idempotents. If $M_{ij} = A_{ij} \cap M$, then

$$A_{ii} = Fe_i \oplus M_{ii} \quad (1 \le i \le n) \quad \text{and} \quad A_{ij} = Fu_{ij} \oplus M_{ij} \quad (i \ne j).$$

In what follows, identity (2.1), relations (2.4) and (4.1) and the fact that $u_{ij} = u_{ji}$ will be frequently used without being mentioned.

Lemma 4.1. *Let M be a $H_n(F)$-module, let $i, j, k \in \{1, \dots, n\}$ be distinct and let $z_{ik} \in M_{ik}$. Then,*

$$4u_{ij}(u_{ij}z_{ik}) = z_{ik}. \tag{4.2}$$

Proof. It follows from $[e_i, u_{ij}, u_{ij}, z_{ik}] = u_{ij}(u_{ij}z_{ik}) - \frac{1}{4}z_{ik}$. □

Lemma 4.2. *Let M be a $H_n(F)$-module, let $i, j, k, l \in \{1, \dots, n\}$ be distinct and let $z_{kl} \in M_{kl}$. Then,*

$$2u_{ij}(u_{jk}z_{kl}) = u_{ik}z_{kl}. \tag{4.3}$$

Proof. It follows from $[u_{ki}, u_{kj}, u_{kj}, z_{kl}] = 4u_{ij}(u_{jk}z_{kl}) - 2u_{ik}z_{kl}$. □

Lemma 4.3. *Let M be a $H_n(F)$-module, let $i, j, k \in \{1, \dots, n\}$ be distinct and let $z_{jk} \in M_{jk}$.*

$$\textit{If} \quad u_{jk}z_{jk} = 0 \quad \textit{then} \quad 2u_{jk}(u_{ik}z_{jk}) = -u_{ij}z_{jk}. \tag{4.4}$$

Proof. It follows from $[e_j, u_{jk}, u_{ik}, z_{jk}] = u_{jk}(u_{ik}z_{jk}) + \frac{1}{2}u_{ij}z_{jk}$. □

Proposition 4.4. *Let M be a $H_n(F)$-module and let $z_{12} \in M_{12}$. Define*

$$z_{1j} = 2u_{j2}z_{12}, \quad \textit{for } j \ge 3 \quad \textit{and} \tag{4.5}$$

$$z_{ij} = 2u_{i1}z_{1j} \quad \textit{for } i, j \ge 2,\ i \ne j. \tag{4.6}$$

Then, for distinct $i, j, k \in \{1, \dots, n\}$, we have

$$2u_{ij}z_{jk} = z_{ik}, \quad \textit{for } k \ge 2, \tag{4.7}$$

$$2u_{ij}z_{kj} = z_{ki}, \quad \textit{for } i, j \ge 2. \tag{4.8}$$

Proof. Definition (4.6) is a particular case ($j = 1$) of (4.7). The other cases are obtained by multiplying (4.6) by $2u_{1i}$ and $2u_{ki}$ ($k \ge 2$):

$$2u_{1i}z_{ij} = 4u_{1i}(u_{1i}z_{1j}) \overset{(4.2)}{=} z_{1j},$$
$$2u_{ki}z_{ij} = 4u_{ki}(u_{1i}z_{1j}) \overset{(4.3)}{=} 2u_{k1}z_{1j} \overset{(4.6)}{=} z_{kj}.$$

In order to obtain relations (4.8) for $k = 1$, we multiply (4.5) by $2u_{2j}$ and $2u_{ij}$ $(i \geq 3)$:

$$2u_{2j}z_{1j} = 4u_{j2}(u_{j2}z_{12}) \overset{(4.2)}{=} z_{12},$$
$$2u_{ij}z_{1j} = 4u_{ij}(u_{j2}z_{12}) \overset{(4.3)}{=} 2u_{i2}z_{12} \overset{(4.5)}{=} z_{1i}. \tag{4.9}$$

The remaining cases $(k \geq 2)$ follow using (4.6) and (4.9) in

$$[e_1, u_{1j}, u_{ki}, z_{1k}] = \tfrac{1}{4}z_{ji} - \tfrac{1}{2}u_{ik}z_{jk}. \quad \square$$

5. Submodules isomorphic to $S_n(F)$

Suppose there exists a nonzero element $v_{12} \in M_{12}$ such that

$$u_{12}v_{12} = 0. \tag{5.1}$$

The aim of this section is to prove that the submodule of M generated by v_{12} is isomorphic to the module $S_n(F)$ of Example 3.2.

Define

$$v_{1j} = 2u_{j2}v_{12}, \quad \text{for } j \geq 3; \tag{5.2}$$
$$v_{ij} = 2u_{i1}v_{1j}, \quad \text{for } i, j \geq 2,\ i \neq j; \tag{5.3}$$
$$v_{j1} = -v_{1j} \quad \text{for } j \geq 2. \tag{5.4}$$

Note that definitions (5.2) and (5.3) are similar to (4.5) and (4.6).

Lemma 5.1. *For the elements v_{ij} defined in (5.1)–(5.4), we have*

$$v_{ji} = -v_{ij} \quad \textit{for every } i, j \in \{1, \ldots, n\},\ i \neq j. \tag{5.5}$$

Proof. The relation holds if $1 \in \{i, j\}$, by definition (5.4).

If $j \geq 3$, then

$$v_{2j} \overset{(5.3)}{=} 2u_{12}v_{1j} \overset{(5.2)}{=} 4u_{12}(u_{j2}v_{12}) \overset{(4.4)}{=} -2u_{1j}v_{12} \overset{(5.3)}{=} -v_{j2}. \tag{5.6}$$

For distinct $i, j \geq 3$, we use (5.3), (4.7) and (5.6) in

$$[e_1, u_{12}, u_{1j}, v_{2i}] = \tfrac{1}{4}v_{ji} + \tfrac{1}{4}v_{ij},$$

which completes the proof. $\square$

Lemma 5.2. *For the elements v_{ij} defined in (5.1)–(5.4), we have*

$$u_{ij}v_{ij} = 0, \quad \textit{for every } i,j \in \{1,\dots,n\},\ i \neq j.$$

Proof. If $\{i,j\} = \{1,2\}$, the relation holds by (5.1) and (5.4). For $j \geq 3$, we have

$$u_{2j}v_{1j} \overset{(5.2)}{=} 2u_{2j}(u_{2j}v_{12}) \overset{(5.4)}{=} -2u_{2j}(u_{2j}v_{21}) \overset{(4.2)}{=} -\tfrac{1}{2}v_{21} \overset{(5.4)}{=} \tfrac{1}{2}v_{12}. \tag{5.7}$$

Using (5.1), (5.2), (5.3), (5.6) and (5.7), we obtain

$$0 = [u_{12}, u_{2j}, u_{2j}, v_{12}] = 2u_{1j}v_{1j} \quad (j \geq 3). \tag{5.8}$$

The remaining cases $(i, j \geq 2,\ i \neq j)$, follow from (5.4), (4.7) and (5.8) in

$$[e_1, u_{1i}, u_{1j}, v_{ij}] = -\tfrac{1}{2}u_{ij}v_{ij}. \quad \square$$

Lemma 5.3. *For the elements v_{ij} defined in (5.1)–(5.4), we have*

$$2u_{ij}v_{jk} = v_{ik}, \quad \textit{for distinct } i,j,k \in \{1,\dots,n\}.$$

Proof. By Proposition 4.4, it only remains to prove the case $k = 1$. For $i, j \geq 2$, $i \neq j$, we have

$$v_{1i} \overset{(4.7)}{=} 2u_{1j}v_{ji} \overset{(5.5)}{=} -2u_{1j}v_{ij} \overset{(5.3)}{=} -4u_{1j}(u_{i1}v_{1j}) \overset{(4.4)}{=} 2u_{ij}v_{1j} \overset{(5.4)}{=} -2u_{ij}v_{j1}.$$

Then, $2u_{ij}v_{j1} = -v_{1i} \overset{(5.4)}{=} v_{i1}$, which concludes the proof. $\square$

By Lemmas 5.1, 5.2 and 5.3, $\{v_{ij} : 1 \leq i < j \leq n\}$ satisfy relations (3.3). As $S_n(F)$ is an irreducible $H_n(F)$-module and $v_{12} \neq 0$, the submodule of M generated by v_{12} is isomorphic to $S_n(F)$. Therefore, we have proved:

Theorem 5.4. *Let M be a $H_n(F)$-module. If there exists a nonzero element $v_{12} \in M_{12}$ such that $u_{12}v_{12} = 0$, then the submodule of M generated by v_{12} is isomorphic to the module $S_n(F)$ of Example 3.2.*

6. Submodules isomorphic to the regular module

Now, suppose there exists a nonzero element $z_{12} \in M_{12}$ such that $u_{12}z_{12} \neq 0$. Let

$$u_{12}z_{12} = \varepsilon_1 + \varepsilon_2, \quad \varepsilon_i \in A_{ii},\ i = 1, 2. \tag{6.1}$$

As in (4.5) and (4.6), define

$$z_{1j} = 2u_{j2}z_{12}\ (j \geq 3) \quad \text{and} \quad z_{ij} = 2u_{i1}z_{1j}\ (i, j \geq 2,\ i \neq j).$$

Moreover, let

$$z_{i1} = 2u_{21}z_{i2} \quad (i \geq 3); \tag{6.2}$$

$$z_{21} = 2u_{23}z_{31}; \tag{6.3}$$

$$\varepsilon_i = e_i(u_{1i}z_{1i}) \quad (i \geq 2). \tag{6.4}$$

We shall prove that $w_{12} = \frac{1}{2}(z_{12} + z_{21})$ generates a submodule isomorphic to the $H_n(F)$-regular module.

Lemma 6.1. *For the elements z_{ij} and ε_i defined above, we have*

$$u_{ij}z_{ij} = \varepsilon_i + \varepsilon_j, \quad \textit{for all } i, j \in \{1, \dots, n\},\ i \neq j. \tag{6.5}$$

Proof. For $i = 1$ and $j = 2$, the Lemma follows from (6.1).

Note that $e_1(u_{12}z_{12}) \overset{(6.1)}{=} \varepsilon_1$ and, if $j \geq 3$, using (4.5), (4.6) and (6.1) in

$$[e_2, u_{1j}, u_{2j}, z_{12}] = \tfrac{1}{2}\big(u_{2j}z_{j2} + \varepsilon_1 - e_2(u_{2j}z_{j2}) - u_{1j}z_{1j}\big)$$

and multiplying the above relation by e_1, we obtain the other cases of identity

$$e_1(u_{1j}z_{1j}) = \varepsilon_1 \quad (j \geq 2). \tag{6.6}$$

As $u_{1j}z_{1j} \in A_{11} + A_{jj}$, using (6.4) and (6.6), we obtain

$$u_{1j}z_{1j} = \varepsilon_1 + \varepsilon_j \quad (j \geq 2). \tag{6.7}$$

Relation

$$u_{ij}z_{ij} = \varepsilon_i + \varepsilon_j \quad (i, j \geq 2, i \neq j) \tag{6.8}$$

is obtained by using (4.6), (4.8) and (6.7) in $[e_1, u_{1i}, u_{ij}, z_{1j}]$. Now, we use (4.7), (6.1) and (6.2) to obtain

$$[e_1, u_{1i}, u_{12}, z_{i2}] = \tfrac{1}{2}\big(u_{i1}z_{i1} - e_1(u_{i1}z_{i1}) - \varepsilon_i\big) \quad \text{and}$$
$$[e_i, u_{1i}, u_{12}, z_{i2}] = \tfrac{1}{2}\big(u_{i1}z_{i1} - e_i(u_{i1}z_{i1}) - \varepsilon_1\big).$$

Comparing these two identities, we obtain

$$e_1(u_{i1}z_{i1}) - \varepsilon_1 = e_i(u_{i1}z_{i1}) - \varepsilon_i \in A_{ii} \cap A_{11} = 0.$$

Therefore, $e_1(u_{i1}z_{i1}) = \varepsilon_1$, $e_i(u_{i1}z_{i1}) = \varepsilon_i$ and

$$u_{i1}z_{i1} = \varepsilon_1 + \varepsilon_i \quad (i \geq 3). \tag{6.9}$$

It only remains to prove the identity for the element z_{21}. First, observe that

$$u_{21}z_{k1} \overset{(6.2)}{=} 2u_{21}(u_{21}z_{k2}) \overset{(4.2)}{=} \tfrac{1}{2}z_{k2} \quad (k \geq 3). \tag{6.10}$$

Then, using (6.3), (6.8), (6.9) and (6.10), we obtain

$$[e_1, u_{12}, u_{23}, z_{31}] = \tfrac{1}{2}\big(u_{12}z_{21} - e_1(u_{12}z_{21}) - \varepsilon_2\big) \quad \text{and}$$
$$[e_2, u_{12}, u_{23}, z_{31}] = \tfrac{1}{2}\big(u_{12}z_{21} - e_2(u_{12}z_{21}) - \varepsilon_1\big).$$

Proceeding as before, we conclude that $u_{12}z_{21} = \varepsilon_1 + \varepsilon_2$. □

Lemma 6.2. *For the elements z_{ij} and ε_i defined in the beginning of this section, we have*

$$4u_{ij}\varepsilon_i = z_{ij} + z_{ji}, \quad \textit{for all } i, j \in \{1, \ldots, n\},\ i \neq j. \tag{6.11}$$

Proof. For $i, j \geq 2$, $i \neq j$, the result follows using (4.6), (4.8) and (6.5) in $[e_1, u_{1i}, u_{ji}, z_{1i}]$. For $i = 1$ and $j \geq 3$, we use (4.5), (4.7), (4.8), (6.2) and (6.5) to obtain

$$[e_1, u_{1j}, u_{2j}, z_{j2}] = 2u_{1j}\varepsilon_j - e_1(u_{1j}\varepsilon_j) - \tfrac{3}{8}z_{1j} - \tfrac{3}{8}z_{j1}. \tag{6.12}$$

Therefore, $2u_{1j}\varepsilon_j - e_1(u_{1j}\varepsilon_j) = \frac{3}{8}(z_{1j} + z_{j1}) \in M_{1j}$. Write $u_{1j}\varepsilon_j = a + b$, with $a \in M_{11}$ and $b \in M_{1j}$. Then, the following element belongs to M_{1j}:

$$2u_{1j}\varepsilon_j - e_1(u_{1j}\varepsilon_j) = 2a + 2b - a - \tfrac{1}{2}b = a + \tfrac{3}{2}b.$$

Then, $a = 0$ and $u_{1j}\varepsilon_j \in M_{1j}$. So, from (6.12) we get

$$4u_{1j}\varepsilon_j = z_{1j} + z_{j1}. \tag{6.13}$$

Analogously, as in (6.12), using $[e_j, u_{1j}, u_{12}, z_{12}] = 2u_{1j}\varepsilon_1 - e_j(u_{1j}\varepsilon_1) - \frac{3}{8}(z_{j1} + z_{1j})$, we obtain

$$4u_{1j}\varepsilon_1 = z_{1j} + z_{j1}.$$

Now, using (4.7), (6.5) and (6.13) in $[e_2, u_{23}, u_{13}, z_{23}]$, it follows that

$$4u_{23}(u_{13}z_{23}) = z_{31}$$

and, multiplying by u_{23} and using (4.2) and (6.3), we get

$$u_{13}z_{23} = u_{23}z_{31} = \tfrac{1}{2}z_{21}. \tag{6.14}$$

Then, we use (4.7), (4.8), (6.3), (6.5) and (6.14) in

$$[e_1, u_{12}, u_{23}, z_{23}] = 2u_{12}\varepsilon_2 - e_1(u_{12}\varepsilon_2) - \tfrac{3}{8}(z_{12} + z_{21}) \quad \text{and}$$
$$[e_2, u_{12}, u_{13}, z_{13}] = 2u_{12}\varepsilon_1 - e_2(u_{12}\varepsilon_1) - \tfrac{3}{8}(z_{21} + z_{12}).$$

As before, we obtain $4u_{12}\varepsilon_2 = z_{12} + z_{21}$ and $4u_{12}\varepsilon_1 = z_{12} + z_{21}$, which concludes the proof. □

Lemma 6.3. *For the elements z_{ij} and ε_i defined in the beginning of this section, the following identities hold*

$$2u_{ij}z_{jk} = z_{ik} \quad \textit{and} \quad 2u_{ij}z_{kj} = z_{ki}, \quad \textit{for distinct } i, j, k \in \{1, \dots, n\}.$$

Proof. Part of the verification of this Lemma was already done in Proposition 4.4, (6.10) and (6.14). Let us prove the remaining cases.

We have

$$u_{32}z_{21} = 2u_{23}(u_{23}z_{31}) \overset{(4.2)}{=} \tfrac{1}{2}z_{31}.$$

Relation

$$2u_{1j}z_{kj} = z_{k1} \quad (j, k \geq 2,\ j \neq k) \tag{6.15}$$

is obtained using (4.6), (4.8), (6.5) and (6.11) in $[e_1, u_{1j}, u_{1k}, z_{1j}]$. Finally, using also (6.15), we get

$$2u_{i1}z_{k1} = z_{ki} \quad (i, k \geq 2,\ i \neq k) \quad \text{and} \quad 2u_{i1}z_{j1} = z_{i1} \quad (i, j \geq 2,\ i \neq j)$$

computing $[e_1, u_{1i}, u_{1i}, z_{ki}]$ and $[e_i, u_{ij}, u_{1i}, z_{ji}]$, respectively. □

Now, for $i, j \in \{1, \dots, n\}$, $i \neq j$, define

$$w_{ij} = \tfrac{1}{2}(z_{ij} + z_{ji}).$$

Note that $w_{12} \neq 0$, since $u_{12}w_{12} = \varepsilon_1 + \varepsilon_2 = u_{12}z_{12} \neq 0$. It is straightforward to verify that the action of $H_n(F)$ on $\{\varepsilon_i, w_{ij} : i, j = 1, \dots, n\}$ is the same as given in (3.2), replacing $\overline{e_i}$ with ε_i and $\overline{u_{ij}}$ with w_{ij}, using Lemmas 6.1, 6.2 and 6.3. Again, as the regular module is irreducible and $w_{12} \neq 0$, the submodule of M generated by w_{12} is isomorphic to the regular $H_n(F)$-module. Also, observe that, in terms of z_{12}, we write $z_{21} = 8u_{23}\big(u_{12}(u_{13}z_{12})\big)$. Therefore, we have proved:

Theorem 6.4. *Let M be a $H_n(F)$-module. If there exists a nonzero element $z_{12} \in M_{12}$ such that $u_{12}z_{12} \neq 0$, then the submodule of M generated by $w_{12} = \frac{1}{2}(z_{12} + z_{21})$, where $z_{21} = 8u_{23}\big(u_{12}(u_{13}z_{12})\big)$, is isomorphic to the regular $H_n(F)$-module.*

7. Irreducible non Jordan modules

As $n \geq 3$, we are under the hypothesis of the results obtained in [12]. The following results are proved in [12], for A a commutative power-associative algebra with $n \geq 3$ orthogonal idempotents $\mathbf{e}_1, \ldots, \mathbf{e}_n$, such that $1 = \mathbf{e}_1 + \cdots + \mathbf{e}_n$, a set of elements $\mathbf{u}_{12}, \ldots, \mathbf{u}_{1n}$ such that $\mathbf{e}_1\mathbf{u}_{1i} = \mathbf{e}_i\mathbf{u}_{1i} = \frac{1}{2}\mathbf{u}_{1i}$ and $\mathbf{u}_{1i}^2 = \mathbf{e}_1 + \mathbf{e}_i$, $i = 2, \ldots, n$ and $A = \bigoplus_{i,j=1}^{n} A_{ij}$ being the Peirce decomposition of A relative to this set of idempotents:

(B1) For $i = 1, \ldots, n$, the subspaces

$$B_i = \{b \in A_{ii} : bA_{ik} \subseteq A_{kk}, \text{ for } k = 1, \ldots, n\}$$

are pairwise isomorphic under the linear isomorphism which sends $b \in B_i$ to $b\mathbf{u}_{ij} \in B_j$, for $i \neq j$, where $\mathbf{u}_{ij} = 2\mathbf{u}_{1i}\mathbf{u}_{1j}$. Moreover, the ideal defined in (2.3) for the idempotent $\mathbf{e}_i + \mathbf{e}_j$ can be obtained in terms of these subspaces as $B_{\mathbf{e}_i+\mathbf{e}_j} = B_i + B_j$.

(B2) The ideal

$$B := \sum_{i \neq j} B_{\mathbf{e}_i+\mathbf{e}_j} = \sum_{i \neq j} B_i + B_j = \bigoplus_{i=1}^{n} B_i \tag{7.1}$$

satisfies $B^2 = 0$ and $B = 0$ if and only if A is a Jordan algebra.

Given M, a unital commutative power associative module for $H_n(F)$, we will consider the split null extension $A = H_n(F) + M$. Taking $\{e_1, \ldots, e_n\}$ to be the orthogonal set of idempotents, the elements $u_{12}, \ldots, u_{1n} \in H_n(H)$ will satisfy the above conditions.

Note that, for every $b \in B_1$, we have, for distinct $i, j \in \{2, \ldots, n\}$,

$$[e_1, u_{1i}, u_{ij}, b] = \tfrac{3}{2}\big(u_{1j}b - u_{ij}(u_{1i}b)\big).$$

Therefore

$$u_{ij}(u_{1i}b) = bu_{1j}, \quad \text{for distinct } i, j \in \{2, \ldots, n\}. \tag{7.2}$$

Lemma 7.1. *Let M be a commutative power-associative module for $H_n(F)$ and let $A = H_n(F)+M$ the split null extension of M. Then, the ideal B defined above for this algebra A is contained in M.*

Proof. Let $b = b_1 + \cdots + b_n \in B$, where $b_i \in B_i$, $i = 1, \ldots, n$. As B is an ideal of A, we have $b_i = e_ib \in B$. Recall that $B_i \subseteq A_{ii} = Fe_i \oplus M_{ii}$. Then, $b_i = \alpha_ie_i + x_i$, with $\alpha_i \in F$ and $x_i \in M_{ii}$. Since $B^2 = 0$, we must have $0 = b_i^2 = \alpha_i^2e_i + \alpha_ix_i$; therefore, $\alpha_i = 0$, for all $i = 1, \ldots, n$. Then $b = x_1 + \cdots + x_n \in M$. □

Proposition 7.2. *Let M be a commutative power-associative module for $H_n(F)$. If M is not a Jordan module then M contains a submodule isomorphic to a direct sum of r copies of the module $M_{n\times 1}(F)$ of Example 3.3, for a convenient positive integer r.*

Proof. If M is not a Jordan module, the ideal B defined in (7.1) for the split null extension $A = H_n(F) + M$ is nonzero. By the previous lemma, $B \subseteq M$. Let $\{b_{11}, \dots, b_{1r}\} \subseteq M_{11}$ be a basis of B_1. For each $i = 2, \dots, n$ and $j = 1, \dots, r$, let

$$b_{ij} = u_{1i}b_{1j}.$$

As mentioned in (B1), $\{b_{i1}, \dots, b_{ir}\}$ is a basis of B_i, for all i.

Let M_j the subspace of M generated by $\{b_{1j}, b_{2j}, \dots, b_{nj}\}$, that is, take the jth element of B_i, for all $i = 1, \dots, r$. Relation (7.2) and the properties of Peirce Decomposition show that M_j is an $H_n(F)$-module isomorphic to the irreducible module $M_{n\times 1}(F)$ of Example 3.3, since the action of $H_n(F)$ on M_j is the same as (3.4). Therefore,

$$B = B_1 \oplus \cdots \oplus B_n = M_1 \oplus \cdots \oplus M_r,$$

is isomorphic to a direct sum of r copies of the module $M_{n\times 1}(F)$. □

Corollary 7.3. *The module $M_{n\times 1}(F)$ given in the Example 3.3 is the only irreducible unital module for the commutative power-associative algebra $H_n(F)$ which is not a Jordan module.*

8. Classification of finite dimensional unital modules

We are ready to prove our main theorem.

Theorem 8.1. *Let F be a field of characteristic different from 2, 3 and 5 and let $n \geq 3$ an integer. Then, there are, up to isomorphisms, three unital irreducible commutative power-associative $H_n(F)$-modules: the regular $H_n(F)$-module, and the modules $S_n(F)$ and $M_{n\times 1}(F)$ of Examples 3.2 and 3.3. Moreover, every finite dimensional unital $H_n(F)$-module is completely reducible.*

Proof. Let M be a finite dimensional unital $H_n(F)$-module and let $M = \bigoplus_{i,j=1}^{n} M_{ij}$ be the Peirce Decomposition of M.

Let B the ideal defined in (7.1) for the split null extension $A = H_n(F) + M$. By Proposition 7.2, if M is not a Jordan module, we have a basis $\{b_{ij} : i = 1, \dots, n; j = 1, \dots, r\}$ of B, with $b_{ij} \in M_{ii}$, $i = 1, \dots, n$, and B isomorphic to the sum of r copies of the irreducible module $M_{n\times 1}(F)$ of Example 3.3. If M is a Jordan module, then $B = 0$ and $r = 0$.

Define $M_{12}^0 = \{m \in M_{12} : u_{12}m = 0\}$ and let $\{v_{12}^{(1)}, \dots, v_{12}^{(t)}\}$ be a basis of M_{12}^0. Complete to a basis $\{v_{12}^{(1)}, \dots, v_{12}^{(t)}, z_{12}^{(1)}, \dots, z_{12}^{(s)}\}$ of M_{12} (we could have t and/or s equal to zero). Then, we have $u_{12}z_{12}^{(k)} \neq 0$, for $k = 1, \ldots, t$. For each k, let $w_{12}^{(k)} = \frac{1}{2}(z_{12}^{(k)} + z_{21}^{(k)})$, where $z_{21}^{(k)} = 8u_{23}\big(u_{12}(u_{13}z_{12}^{(k)})\big)$. Then $I = \{v_{12}^{(1)}, \ldots, v_{12}^{(t)}, w_{12}^{(1)}, \ldots, w_{12}^{(s)}\}$ is still a basis of M_{12}, since $x_{12}^{(k)} = \frac{1}{2}(z_{12}^{(k)} - z_{21}^{(k)}) \in M_{12}^0$ and $w_{12}^{(k)} = z_{12}^{(k)} + x_{12}^{(k)}$.

By Theorem 6.4, $w_{12}^{(k)}$ generates a submodule isomorphic to the regular $H_n(F)$-module. Let $w_{ij}^{(k)} \in M_{ij}$ and $\varepsilon_i^{(k)} \in M_{ii}$, $i, j = 1, \ldots, n$, be the elements produced starting with $w_{12}^{(k)}$ as in the beginning of Section 6, for $k = 1, \ldots, s$. Also, by Theorem 5.4, $v_{12}^{(i)}$ generates a submodule isomorphic to $S_n(F)$. As well, let $v_{ij}^{(l)} \in M_{ij}$, $i, j = 1, \ldots, n$, be the elements obtained from $v_{12}^{(l)}$ as in the beginning of Section 5, for $l = 1, \ldots, t$. We claim that $C = \{v_{ij}^{(l)}, w_{ij}^{(k)}, \varepsilon_i^{(k)} : i, j = 1, \ldots, n; l = 1, \ldots, t; k = 1, \ldots, s\}$ is a linearly independent set. This fact follows by observing that $C \cap M_{ii} = \{\varepsilon_i^{(1)}, \ldots, \varepsilon_i^{(s)}\}$ and $C \cap M_{ij} = \{v_{ij}^{(1)}, \ldots, v_{ij}^{(t)}, w_{ij}^{(1)}, \ldots, w_{ij}^{(s)}\}$, for $i \neq j$, and any linear combination of elements in these sets can be transformed into a linear combination of elements in I by multiplying for convenient elements u_{1i} and/or u_{2j}, depending if $i, j \in \{1, 2\}$ or not, using Lemmas 5.3 and 6.3 (for instance, for a linear combination of elements in $C \cap M_{34}$, multiply by u_{13} and, then, by u_{24}; for $C \cap M_{23}$, multiply only by u_{13}; for $C \cap M_{33}$, multiply by u_{13} and, then, by u_{23}).

The same procedure used to show that $C \cap M_{ij}$ is a linearly independent set shows that this set is a basis of M_{ij}: take some $m \in M_{ij}$. By multiplying for convenient elements u_{1i} and/or u_{2j} we get an element in M_{12} which has basis I. Then, this element can be written as linear combination of $v_{12}^{(l)}$ and $w_{12}^{(k)}$, $l = 1, \ldots, t$ and $k = 1, \ldots, s$. Multiplying again by the same u_{1i} and/or u_{2j} in the reverse order (if there are two) and using also Lemma 4.1, we obtain, on the one hand, a scalar multiple of m and, on the other hand, a linear combination of elements in $C \cap M_{ij}$. For instance, if $m \in M_{23}$ then $\frac{1}{4}m = u_{13}(\underbrace{u_{13}m}_{\in M_{12}})$; if $m \in M_{34}$ then $\frac{1}{16}m = u_{13}\Big(u_{24}\big(\underbrace{u_{24}(u_{13}m)}_{\in M_{12}}\big)\Big)$.

Finally, let $m \in M_{ii}$. Then, for each $j = 1, \ldots, n$, $j \neq i$, we have $u_{ij}m \in M_{ij} + M_{jj}$. So, write

$$u_{ij}m = z_{ij} + y_{ij} + x_j, \tag{8.1}$$

with $z_{ij} \in \operatorname{span}\{w_{ij}^{(k)}, 1 \leq k \leq s\}$, $y_{ij} \in \operatorname{span}\{v_{ij}^{(l)}, 1 \leq l \leq t\}$ and $x_j \in M_{jj}$. First, we will show that $y_{ij} = 0$. In order to prove this statement consider $j \neq k$ (different from i) and write

$$y_{ij} = \sum_{l=1}^{t} \alpha_l v_{ij}^{(l)} \quad \text{and} \quad y_{ik} = \sum_{l=1}^{t} \beta_l v_{ik}^{(l)}.$$

On the one hand, using $[e_j, u_{ij}, u_{ik}, m] = 0$ we obtain that $\alpha_l + \beta_l = 0$ and, on the other hand, from $e_i[e_j, u_{ij}, u_{jk}, m] = 0$, we conclude that $\alpha_l - \beta_l = 0$. Then, $\alpha_l = 0\,(= \beta_l)$, for each $l = 1, \ldots, t$. Consequently, $y_{ij} = 0$ and we can rewrite (8.1) as

$$u_{ij}m = \sum_{k=1}^{s} \gamma_k w_{ij}^{(k)} + x_j,$$

where $\gamma_k \in F$. Let $n = m - \sum_{k=1}^{s} 2\gamma_i \varepsilon_i^{(k)}$. Then,

$$u_{ij}n = u_{ij}m - \sum_{k=1}^{s} \gamma_i w_{ij}^{(k)} = x \in M_{jj}, \text{ for } j = 1, \dots, n.$$

Therefore, $n \in B_i$ and $m = n + \sum_{k=1}^{s} 2\gamma_i \varepsilon_i^{(k)} \in B_i + \operatorname{span}(C \cap M_{ii})$, that is $M_{ii} = B_i + \operatorname{span}(C \cap M_{ii})$. Moreover, as $u_{ij}B_i \subseteq B_j \subseteq M_{jj}$ and $u_{ij}\varepsilon_i^{(k)} \in M_{ij}$, for $j \neq i$ and $k = 1, \dots, s$, we conclude that $B_i \cap \operatorname{span}(C \cap M_{ii}) = 0$. This means that $\{b_{i1}, \dots, b_{ir}\} \cup (C \cap M_{ii})$ is a basis of M_{ii}.

We have completely described the Peirce components of M, producing a basis of each one of these components, in a way that they also form bases for the irreducible submodules of M: we have obtained that M direct sum of r copies of $M_{n\times 1}(F)$, t copies of $S_n(F)$ and s copies of the regular $H_n(F)$-module. □

As a consequence, we reobtain the classification of irreducible unital Jordan modules for $H_n(F)$ due to N. Jacobson ([7]).

Corollary 8.2. *Let $n \geq 3$ a integer. Then, there exist exactly two nonisomorphic irreducible unital Jordan modules for $H_n(F)$: the regular $H_n(F)$-module and $S_n(F)$, the module of skew-symmetric matrices over F.*

The description of irreducible unital commutative power-associative modules for $H_2(F)$ is quite different. The structure of these modules will be described in another paper.

Data availability

No data was used for the research described in the article.

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